

# Separability and entanglement in $\mathcal{C}^2 \otimes \mathcal{C}^2 \otimes \mathcal{C}^N$ composite quantum systems

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We investigate separability and entanglement of mixed states in  $\mathcal{C}^2 \otimes \mathcal{C}^2 \otimes \mathcal{C}^N$  three party quantum systems. We show that all states with positive partial transposes that have rank  $\leq N$  are separable. For the 3 qubit case ( $N = 2$ ) we prove that all states  $\rho$  that have positive partial transposes and rank 3 are separable. We provide also constructive separability checks for the states  $\rho$  that have the sum of the rank of  $\rho$  and the ranks of partial transposes with respect to all subsystems smaller than  $15N-1$ .

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## I. INTRODUCTION

In the recent years it became clear that entanglement is one of the most important ingredients of the quantum information processing. While in the early age of quantum mechanics, entanglement was associated with "paradoxes" of quantum mechanics [1,2], in the last decade of the last century it has been discovered that entanglement plays an essential role in fundamental applications of quantum mechanics to information processing (cf. [3–5]). While the characterization of separable and entangled pure states of bipartite systems is quite well understood (cf. [6]), it is not the case for mixed states. In the last four years, however, a lot of progress has been achieved in our understanding of the separability and entanglement problem for bipartite systems (cf. [7]). The first major step was the proper definition of separable and entangled states formulated by Werner [8]. The next milestone was the discovery by Peres [9] of the fact that all separable states are must necessarily have a positive partial transpose [10]. Soon after Horodeckis [11] have shown the Peres criterium provides also a sufficient condition for separability in two qubit ( $2 \times 2$ ) and one qubit one qutrit ( $2 \times 3$ ) systems. Subsequently, P. Horodecki [12] has constructed the first examples of the, so called, bound entangled states, i.e. the first examples of the entangled states with positive partial transpose (PPT ES). This discovery has stimulated great interest in the studies of properties of PPT ES. Some of the most important results, in particular coming from the Horodecki family, IBM group, and Innsbruck–Hannover collaboration are described in Refs. [7].

More recently, considerable interest has been devoted to multiparty entanglement [13]. The first papers on 3 qubit states led to the discovery of the, so called, GHZ states [14], which are particularly suited to study the

break down of the Bell like inequalities in quantum mechanics [6]. Three party (and multiparty, in general) entanglement of the GHZ type can allow for interesting applications, such as for instance quantum secret sharing [15], and many experimental groups have recently tried to generate such states [16].

Theoretical studies of the structure of multiparty entangled states has just started [13]. First of all, pure state entanglement has been investigated. An important direction of research was here initiated by Ref. [17]. In this paper Linden and Popescu have studied whether a given quantum state can be transformed into another one using local unitary (or at least non-unitary invertible) transformations. Such a geometric approach calls for studies of invariants of local unitary and local non-unitary invertible transformations, and leads elegantly and naturally to the concepts of Schmidt coefficients [6], and Schmidt number [18] for pure states in bipartite systems.

This approach and concepts can be generalized to the case of 3 qubit systems and in general for mixed states, but it is by no means an easy task. In particular, as pointed out in Ref. [19], in both cases one expects various, locally not equivalent kinds of entanglement to arise. Very recently the concept of Schmidt coefficients (i.e. invariants of the local unitary transformations) has been formulated for 3 qubit systems [20,21]. The other approach (based on the investigations of local non-unitary invertible operations) has been followed by the Innsbruck group [22]. Dür *et al.* were able to show that there are essentially 3 types of entanglement of pure states: bipartite entanglement, *W*-class entanglement, and *GHZ*-class entanglement. The are ways of characterizing the 3-qubit entanglement with the help of a, so called, tangle [23], and other local invariants [20,21]. Numerous studies of various types of multiparty entanglement of pure states and various interesting examples of it have been conducted in the recent years [24].

At this point it is worth mentioning that while for the pure states in bipartite systems it is possible to quantify, or better to say to characterize the entanglement in the canonical way [19], this is not necessarily the case for the mixed states. The studies of entanglement and separability in mixed states of a three party system has just begun. Among the recent results it is worth mentioning the demonstration of separability of the states that differ not much from fully chaotic state [25], the construction of entangled states that have all partial transposes positively defined, employing the concept of unextendible product basis [26], the classification of multi-qubit states based

on the separability and distillability properties of certain partitions, [27], the generalization of the concept of mean Schmidt number to the case of multiparty systems [28], a formulation of the necessary and sufficient conditions for separability in term of linear maps [29], studies of the properties of relative entropy in multiparty systems [30], and studies of states symmetric with respect to trilateral unitary rotations [31]. We have presented recently [32] a classification of mixed states for 3 qubit systems into the separable class, the bipartite, the  $W$ -, and the  $GHZ$ -classes of states. Following the Refs. [33] we constructed canonical form of entanglement witnesses for each class, and discussed their optimization.

In this paper we consider entanglement and separability of mixed states in  $\mathcal{C}^2 \otimes \mathcal{C}^2 \otimes \mathcal{C}^N$  three party quantum systems. Such systems are of practical interests since i) for  $N = 2$  they reduce to the intensively studied 3 qubit systems; ii) for  $N$  large, they can be used to describe two qubits interacting via a "bus" mode; this is how the quantum gates can be realized in the quantum computer model based on cold trapped ions [34,35].

This paper generalizes the results obtained by us earlier for the case  $2 \times N$  [36] and  $M \times N$  [37] systems. We use here the same mathematical tools that have been developed in our earlier work [38,39], i.e. the method of subtracting from a given state  $\rho$  projectors on product states keeping the remainder, as well as its partial transpose ( $-s$ ) positively definite

The paper is organized as follows. In section 2 we demonstrate that all states in  $\mathcal{C}^2 \otimes \mathcal{C}^2 \otimes \mathcal{C}^N$  systems with positive partial transposes that have rank  $\leq N$  are separable. This section is divided into 3 subsections, and the main result is presented in the last subsection. In the first subsection we present the canonical form of the investigated states; in the second one we prove an important Lemma that states that for the 3 qubit case ( $N = 2$ ) all states  $\rho$  that have positive partial transposes and rank 3 are separable. In the section 3 we discuss constructive criteria and separability checks for the states  $\rho$  that have the sum of the rank of  $\rho$  and the ranks of partial transposes with respect to all subsystems smaller equal than  $15N - 1$ . We discuss here the concept of the *edge* states, i.e. those from which no projector on a product state can be subtracted without loosing either the positivity, or the PPT property. We discuss here also the methods of constructing the, so called, entanglement witnesses, and their canonical form. In section 4 we specify the previous results for the case  $N = 2$ , and we provide constructive separability checks for the states  $\rho$  that have the sum of the rank of  $\rho$  and the ranks of partial transposes with respect to all subsystems smaller equal than 29.

In this paper we denote by  $R(\rho)$ ,  $K(\rho)$ ,  $r(\rho)$  and  $k(\rho)$  the range, the kernel, the rank, the dimension of the kernel of  $\rho$ , respectively. Also,  $|\hat{e}\rangle$  will denote a vector orthogonal to  $|e\rangle$ . The symbol  $\text{diag}[\sigma_1, \sigma_2, \dots]$  denotes a matrix with diagonal blocks  $\sigma_1, \sigma_2, \dots$

## II. PPT STATES OF RANK $N$ IN $\mathcal{C}^2 \otimes \mathcal{C}^2 \otimes \mathcal{C}^N$ SYSTEMS

### A. Generic form of the rank $N$ PPT states

In this section we will derive the canonical form of the separable states in  $\mathcal{C}^2 \otimes \mathcal{C}^2 \otimes \mathcal{C}^N$  with  $r(\rho) = N$ . The canonical form will allow for an explicit decomposition of a given state in terms of convex sum of projectors on product vectors. In the following the three parties will be called Alice, Bob and Charlie. We begin with the following Lemma:

**Lemma 1 :** *Every PPT state  $\rho$  in  $\mathcal{C}^2 \otimes \mathcal{C}^2 \otimes \mathcal{C}^N$  with  $r(\rho) = N$ , such that in some local basis  $(|0_A\rangle, |1_A\rangle)$  for Alice,  $(|0_B\rangle, |1_B\rangle)$  for Bob,  $(|0_C\rangle, \dots, |N-1_C\rangle)$  for Charlie) without loosing the generality we have  $r(\langle 1_A, 1_B | \rho | 1_A, 1_B \rangle) = N$ , can be transformed using a reversible local operation to the following canonical form:*

$$\begin{aligned} \rho &= \sqrt{D} \begin{pmatrix} B^\dagger C^\dagger C B & B^\dagger C^\dagger C & B^\dagger C^\dagger B & B^\dagger C^\dagger \\ C^\dagger C B & C^\dagger C & C^\dagger B & C^\dagger \\ B^\dagger C B & B^\dagger C & B^\dagger B & B^\dagger \\ C B & C & B & 1 \end{pmatrix} \sqrt{D} \\ &= \sqrt{D} \begin{pmatrix} B^\dagger C^\dagger \\ C^\dagger \\ B^\dagger \\ 1 \end{pmatrix} (C B \ C \ B \ 1) \sqrt{D}, \end{aligned} \quad (1)$$

where  $[B, B^\dagger] = [C, C^\dagger] = [C, B] = [C, B^\dagger] = 0$  and  $D = D^\dagger$ ;  $B, C$  and  $D$  are operators acting in the Charlie's space.

**Proof:** The state  $\rho$  can be always written in the considered basis as:

$$\rho = \begin{pmatrix} E_1 & E_5 & E_6 & E_7 \\ E_5^\dagger & E_2 & E_8 & E_9 \\ E_6^\dagger & E_8^\dagger & E_3 & E_{10} \\ E_7^\dagger & E_9^\dagger & E_{10}^\dagger & E_4 \end{pmatrix},$$

where  $E$ 's are  $N \times N$ -matrices, and  $r(E_4) = N$ . After the projection  $\tilde{\rho} = \langle 1_A | \rho | 1_A \rangle$  we obtain the reduced state

$$\tilde{\rho} = \begin{pmatrix} E_3 & E_{10} \\ E_{10}^\dagger & E_4 \end{pmatrix}.$$

After performing a reversible local non-unitary "filtering"  $\frac{1}{\sqrt{E_4}}$  on Charlie's side the matrix  $\tilde{\rho}$  can be written as:

$$\tilde{\rho} = \begin{pmatrix} A & B^\dagger \\ B & 1 \end{pmatrix}.$$

This matrix is obviously positive, i.e. can be represented as [37]  $\tilde{\rho} = \Sigma + \text{diag}[\Delta, 0]$ , where  $\Delta = A - B^\dagger B$ ,

$$\Sigma = \begin{pmatrix} B^\dagger B & B^\dagger \\ B & 1 \end{pmatrix}.$$

The matrix  $\tilde{\rho}$  muß has the rank  $N$ . We observe that  $\Sigma$  has also the range  $N$ , and possesses  $N$  vectors in its kernel  $|\phi_f\rangle = |1\rangle|f\rangle - |2\rangle B|f\rangle$ . We will show that  $\Delta = 0$ .

Using the fact that  $\tilde{\rho} \geq 0$ , we observe that  $\Delta \geq 0$ . But, since  $r(\tilde{\rho}) = r(\Sigma)$ , the ranges of the the matrices must fulfill  $R(\tilde{\rho}) = R(\Sigma) \supseteq R(\text{diag}[\Delta, 0])$ , so that the corresponding kernels fulfill  $K(\text{diag}[\Delta, 0]) \supseteq K(\Sigma)$ . The kernel  $K(\Sigma)$  is spanned by the vectors of the form  $|\phi_f\rangle = |1\rangle|f\rangle - |2\rangle B|f\rangle$ , where  $|f\rangle$  is arbitrary, for which  $\langle \phi_f | \text{diag}[\Delta, 0] | \phi_f \rangle = 0$  must hold also. This means, however, that  $\Delta|f\rangle = 0$  for all  $|f\rangle$ , and thus  $\Delta = 0$ .

The fact that  $B$  is a normal operator follows from the fact that  $\tilde{\rho}^{t_A}$  must be positively definite. This condition implies that  $BB^\dagger - B^\dagger B \geq 0$ . The latter positive operator has, however, the trace zero, and must therefore vanish, i.e.  $[B, B^\dagger] = 0$ .

Similarly, if we consider the projection  $\langle 1_B | \rho | 1_B \rangle$ , for the same reasons as above we conclude that the resulting matrix

$$\tilde{\rho} = \begin{pmatrix} C^\dagger C^\dagger & C^\dagger \\ C & 1 \end{pmatrix},$$

with  $[C, C^\dagger] = 0$ . Summarizing, after performing a local filtering operation  $\frac{1}{\sqrt{E_4}}$  we can bring the matrix  $\rho$  to the form:

$$\rho = \begin{pmatrix} E_1 & E_5 & E_6 & E_7 \\ E_5^\dagger & C^\dagger C & E_8 & C^\dagger \\ E_6^\dagger & E_8^\dagger & B^\dagger B & B \\ E_7^\dagger & C^\dagger & B & 1 \end{pmatrix}.$$

Now, the matrix  $\rho$  possesses as kernel vectors  $|10\rangle|f\rangle - |11\rangle B|f\rangle$  and  $|01\rangle|g\rangle - |11\rangle C|g\rangle$  for all  $|f\rangle, |g\rangle$  from the Charlie's space. This implies that we must have  $E_8 = C^\dagger B$ ,  $E_6 = E_7 B$  and  $E_5 = E_7 C$ . The matrix  $\rho$  has thus the form:

$$\rho = \begin{pmatrix} E_1 & E_7 C & E_7 B & E_7 \\ C^\dagger E_7^\dagger & C^\dagger C & C^\dagger C & C^\dagger \\ E_7^\dagger B & E_7^\dagger C & B^\dagger B & B^\dagger \\ E_7^\dagger & C & B & 1 \end{pmatrix}.$$

In the next step we consider its partial transpose with respect to Alice given by:

$$\rho^{t_A} = \begin{pmatrix} E_1 & E_7 C & B^\dagger E_7^\dagger & B^\dagger C \\ C^\dagger E_7^\dagger & C^\dagger C & E_7^\dagger & C^\dagger \\ E_7 B & E_7 & B^\dagger B & B^\dagger \\ C^\dagger B & C^\dagger & B & 1 \end{pmatrix}.$$

Since partial transpose with respect to Alice is positive and does not change  $\langle 1_A | \rho | 1_A \rangle$ , the vectors  $|10\rangle|f\rangle - |11\rangle B|f\rangle$  should remain in the kernel. This implies the equality  $E_7 = B^\dagger C^\dagger$ , and the following form of  $\rho$ :

$$\rho = \begin{pmatrix} E_1 & B^\dagger C^\dagger C & B^\dagger C^\dagger B & B^\dagger C^\dagger \\ C^\dagger C B & C^\dagger C & C^\dagger B & C^\dagger \\ B^\dagger C B & B^\dagger C & B^\dagger B & B^\dagger \\ C B & C & B & 1 \end{pmatrix}.$$

The above form can be rewritten as

$$\rho = \begin{pmatrix} B^\dagger C^\dagger C B & B^\dagger C^\dagger C & B^\dagger C^\dagger B & B^\dagger C^\dagger \\ C^\dagger C B & C^\dagger C & C^\dagger B & C^\dagger \\ B^\dagger C B & B^\dagger C & B^\dagger B & B^\dagger \\ C B & C & B & 1 \end{pmatrix} + \text{diag}[\tilde{\Delta}, 0, 0, 0],$$

where  $\Delta = E_1 - B^\dagger C^\dagger C B$ . Using the short hand notation we get

$$\rho = \begin{pmatrix} B^\dagger C^\dagger \\ C^\dagger \\ B^\dagger \\ 1 \end{pmatrix} (C B \ C \ B \ 1) + \text{diag}[\tilde{\Delta}, 0, 0, 0].$$

The first term in  $\rho$  is PPT and has the following  $3N$  vectors in the kernel:

$$\begin{aligned} |\psi\rangle &= |00\rangle|f\rangle + |11\rangle C B|f\rangle \\ |\phi\rangle &= |01\rangle|g\rangle + |11\rangle C|g\rangle \\ |\chi\rangle &= |10\rangle|h\rangle + |11\rangle B|h\rangle, \end{aligned}$$

for arbitrary  $|f\rangle, |g\rangle$  and  $|h\rangle$ . Similarly as above, this means that as in the case of  $\Delta$ , the matrix  $\tilde{\Delta}$  must vanish. This provides us with the final form of  $\rho$ :

$$\rho = \begin{pmatrix} B^\dagger C^\dagger C B & B^\dagger C^\dagger C & B^\dagger C^\dagger B & B^\dagger C^\dagger \\ C^\dagger C B & C^\dagger C & C^\dagger B & C^\dagger \\ B^\dagger C B & B^\dagger C & B^\dagger B & B^\dagger \\ C B & C & B & 1 \end{pmatrix} \quad (2)$$

$$= \begin{pmatrix} B^\dagger C^\dagger \\ C^\dagger \\ B^\dagger \\ 1 \end{pmatrix} (C B \ C \ B \ 1). \quad (3)$$

It remains only to prove the commutation relations  $[B, C] = [B, C^\dagger] = 0$ . This follows from the positivity of all partial transposes of  $\rho$ . In particular,  $\rho^{t_A}$  is:

$$\rho^{t_A} = \begin{pmatrix} B^\dagger C \\ C \\ B^\dagger \\ 1 \end{pmatrix} (C^\dagger B \ C^\dagger \ B \ 1), \quad (4)$$

which is obviously positive definite.

In contrast,  $\rho^{t_B}$  can be written as:

$$\rho^{t_B} = \begin{pmatrix} B^\dagger C^\dagger C B & C^\dagger C B & B^\dagger C^\dagger B & C^\dagger B \\ B^\dagger C^\dagger C & C^\dagger C & B^\dagger C^\dagger & C^\dagger \\ B^\dagger C B & C B & B^\dagger B & B \\ B^\dagger C & C & B^\dagger & 1 \end{pmatrix}. \quad (5)$$

Because of its positivity, the matrix  $\rho^{t_B}$  must possess the kernel vector  $|01\rangle|g\rangle - |11\rangle C|g\rangle$ , which implies that  $[C, B] = 0$ . The matrix  $\rho^{t_B}$  can be then written as:

$$\rho^{t_B} = \begin{pmatrix} C^\dagger B \\ C^\dagger \\ B \\ 1 \end{pmatrix} (B^\dagger C \ C \ B^\dagger \ 1), \quad (6)$$

which implies automatically the positivity. It remains finally to consider  $\rho^{t_{AB}}$ . The latter can be written as:

$$\rho^{t_{AB}} = \begin{pmatrix} B^\dagger C^\dagger C B & C^\dagger C B & B^\dagger C B & C B \\ B^\dagger C^\dagger C & C^\dagger C & B^\dagger C & C \\ B^\dagger C^\dagger B & C^\dagger B & B^\dagger B & B \\ B^\dagger C^\dagger & C^\dagger & B^\dagger & 1 \end{pmatrix}. \quad (7)$$

From the positivity of  $\rho^{t_{AB}}$  follows that  $|10\rangle - |11\rangle B^\dagger |f\rangle$  is a kernel vector, so that  $[B^\dagger, C] = 0$  must hold. This in turn allows to write  $\rho^{t_{AB}}$  as:

$$\rho^{t_{AB}} = \begin{pmatrix} C B \\ C \\ B \\ 1 \end{pmatrix} \begin{pmatrix} B^\dagger C^\dagger & C^\dagger & B^\dagger & 1 \end{pmatrix}. \quad (8)$$

Again, this form assures positive definiteness, and concludes the proof of the Lemma.  $\square$

Now, we are in the position to prove:

**Lemma 2 :** *A PPT-state  $\rho$  in  $\mathcal{C}^2 \otimes \mathcal{C}^2 \otimes \mathcal{C}^N$ , whose rank  $r(\rho) = N$ , and for which there exists a product basis  $|e_A, f_B\rangle$ , such that  $r(\langle e_A, f_B | \rho | e_A, f_B \rangle) = N$ , is separable.*

**Proof:** The state  $\rho$  can be written according to the Lemma (1) as

$$\rho = \begin{pmatrix} B^\dagger C^\dagger \\ C^\dagger \\ B^\dagger \\ 1 \end{pmatrix} \begin{pmatrix} C B & C & B & 1 \end{pmatrix}. \quad (9)$$

Since all operators commute, they have to have common eigenvectors  $|f_n\rangle$ , with eigenvalues  $b_n, c_n$ , respectively, and

$$\langle f_n | \rho | f_n \rangle = \begin{pmatrix} b_n^* c_n^* \\ c_n^* \\ b_n^* \\ 1 \end{pmatrix} \begin{pmatrix} c_n b_n & c_n & b_n & 1 \end{pmatrix}. \quad (10)$$

This is, however, a product vector in Alice's and Bob's spaces. We can thus write  $\rho$  as  $\rho = \sum_{n=1}^N |\psi_n\rangle\langle\psi_n| \otimes |\phi_n\rangle\langle\phi_n| \otimes |f_n\rangle\langle f_n|$ . Because the local transformations used above were reversible, we can now apply their inverses and obtain a decomposition of the initial state  $\rho$  in a sum of projectors onto product vectors. This proves separability of  $\rho$ , and the Lemma.  $\square$

From the Lemma 1 and 2 we conclude that in order to prove that PPT states  $\rho$  supported on  $\mathcal{C}^2 \otimes \mathcal{C}^2 \otimes \mathcal{C}^N$  with  $r(\rho) = N$  are separable, it is enough to show that one can find a product basis such that  $r(\langle e_A, f_B | \rho | e_A, f_B \rangle) = N$ . We will accomplish the proof in another way. Instead, we will prove the separability directly, and the desired canonical form of  $\rho$  will be a consequence of that. To this aim we will use the results of Ref. [36], and the following theorem from Ref. [37].

**Theorem 1 :** *For all PPT states  $\rho$  that are supported on  $M \times N$ -space ( $M \leq N$ ), and that have rank  $N$ , there exists a product basis such that  $r(\langle 1_A | \rho | 1_A \rangle) = N$  and  $\rho$  is separable and has the form*

$$\rho = \sum_{i=1}^N |e_i, b_i\rangle\langle e_i, b_i| \quad (11)$$

where  $|b_i\rangle$  are linearly independent. Additionally, the above decomposition is unique.

This theorem can be used to prove the following Lemma:

**Lemma 3 :** *Any PPT state  $\rho$  supported on  $\mathcal{C}^2 \otimes \mathcal{C}^2 \otimes \mathcal{C}^N$  with  $N \geq 4$ , for which  $r(\rho) = N$ , is separable, and obeys assumptions of Lemma 1.*

**Proof:** A  $\mathcal{C}^2 \otimes \mathcal{C}^2 \otimes \mathcal{C}^N$ -system can be regarded as a  $\mathcal{C}^4 \otimes \mathcal{C}^N$ -system. From the theorem 1 we obtain that:

$$\rho = |\psi_{AB_1}\rangle\langle\psi_{AB_1}| \otimes |C_1\rangle\langle C_1| + \sum_{i=2}^N |\psi_{AB_i}, C_i\rangle\langle\psi_{AB_i}, C_i|. \quad (12)$$

Note, however, that we can find now a vector  $|C\rangle$  in Charlie's space, so that  $\langle C | \rho | C \rangle \sim |\psi_{AB_1}\rangle\langle\psi_{AB_1}|$ . Because the state  $\rho$  has a PPT property with respect to all partitions,  $|\psi_{AB_1}\rangle\langle\psi_{AB_1}|$  must be PPT with respect to Alice or Bob, i.e. it must be a product state. This observation concerns all projectors that enter the convex sum (12), so that we conclude that  $\rho$  is separable. From (12) it follows directly that  $\rho$  can be projected onto  $|1_A, 1_B\rangle$ , so that  $r(\langle 1_A, 1_B | \rho | 1_A, 1_B \rangle) = N. \square$

## B. Separability of states with rank $\leq 3$ in $\mathcal{C}^2 \otimes \mathcal{C}^2 \otimes \mathcal{C}^2$

Now we have to consider the cases  $N = 2, 3$ . The following Corollary and Lemma deal with the case  $N = 2$ :

**Corollary 1 :** *Any PPT state  $\rho$  supported on  $\mathcal{C}^2 \otimes \mathcal{C}^2 \otimes \mathcal{C}^2$  and such that  $r(\rho) = 2$ , has a product vector  $|e, f, g\rangle$  in its kernel.*

**Proof:** The vector  $|e, f, g\rangle$  belongs to the kernel iff it is orthogonal to the range, i.e. iff it is orthogonal to the two vectors  $\{|\psi_1\rangle, |\psi_2\rangle\}$ , which span the range of  $\rho$ . We can choose  $|e\rangle$  arbitrary and set  $|f\rangle = |0\rangle + \alpha|1\rangle$ , so that we obtain two equations:

$$(\langle\psi_i|e, 0\rangle + \alpha\langle\psi_i|e, 1\rangle)|g\rangle = 0 \quad (13)$$

We treat these equations as linear homogeneous equations for  $|g\rangle$ ; they have nontrivial solutions if the corresponding determinant of a  $2 \times 2$  matrix vanishes. This gives a quadratic equation for  $\alpha$ , which has always at least one solution.  $\square$

**Lemma 4 :** Any PPT state  $\rho$ , supported on  $\mathcal{C}^2 \otimes \mathcal{C}^2 \otimes \mathcal{C}^2$ , and such that  $r(\rho) = 2$ , is separable (compare [40]).

**Proof:** If  $|e_A, f_B, g_C\rangle$  is in the kernel of  $\rho$ , then PPT property implies that also  $\rho^{t_A}|e_A^*, f_B, g_C\rangle = 0$ . We obtain then that  $\langle \hat{e}_A^* | \rho^{t_A} | e_A^*, f_B, g_C \rangle = 0$ , where  $|\hat{e}_A\rangle$  is orthogonal to  $|e_A\rangle$ . This equation is equivalent to  $\langle e_A | \rho | \hat{e}_A, f_B, g_C \rangle = 0$ . This, however, means that  $\rho | \hat{e}_A, f_B, g_C \rangle = | \hat{e}_A, \psi_{BC} \rangle$ , where  $|\psi_{BC}\rangle$  a vector in Bob's and Charlie's space. Now, according to the Lemma 2 of Ref. [36] which deals with  $\mathcal{C}^2 \otimes \mathcal{C}^N$ -systems, we can subtract the projector  $|\hat{e}_A, \psi_{BC}\rangle\langle \hat{e}_A, \psi_{BC}|$  from  $\rho$ , so that  $\tilde{\rho} = \rho - \frac{1}{\langle \hat{e}_A, \psi_{BC} | \rho^{-1} | \hat{e}_A, \psi_{BC} \rangle} |\hat{e}_A, \psi_{BC}\rangle\langle \hat{e}_A, \psi_{BC}|$  is positive, has rank 1, i.e. is a projector. Since it has the PPT property with respect to Alice's system, it must be separable with respect to  $A-BC$  partition. In general, we can thus write  $\rho = \tilde{\Lambda} |\tilde{e}_A\rangle\langle \tilde{e}_A| \otimes |\tilde{\psi}_{BC}\rangle\langle \tilde{\psi}_{BC}| + \Lambda |\hat{e}_A, \psi_{BC}\rangle\langle \hat{e}_A, \psi_{BC}|$ . Projecting onto  $|e_A\rangle$  we get  $\langle e_A | \rho | e_A \rangle \propto |\psi_{BC}\rangle\langle \psi_{BC}|$ . Since  $\rho$  has the PPT property with respect to all partitions, the projector  $|\tilde{\psi}_{BC}\rangle\langle \tilde{\psi}_{BC}|$  must project onto a product vector. The same can be of course said about  $|\psi_{BC}\rangle\langle \psi_{BC}|$ , since the projection onto  $|\hat{e}_A\rangle\langle \hat{e}_A|$  gives  $\langle \hat{e}_A | \rho | \hat{e}_A \rangle \propto |\psi_{BC}\rangle\langle \psi_{BC}|$ , which implies that  $|\psi_{BC}\rangle\langle \psi_{BC}|$  is a product state and concludes the proof.  $\square$

Now we have still to prove the case  $N = 3$ . Before we do that, however, we need one more Corollary and Lemma concerning the case  $N = 2$ :

**Corollary 2 :** Any PPT state  $\rho$ , supported on  $\mathcal{C}^2 \otimes \mathcal{C}^2 \otimes \mathcal{C}^2$ , such that  $r(\rho) = 3$ , has a product vector  $|e, f, g\rangle$  in the kernel.

**Proof:** Let  $\rho$  be PPT-state in  $\mathcal{C}^2 \otimes \mathcal{C}^2 \otimes \mathcal{C}^2$ . It can be regarded as a  $\mathcal{C}_A^2 \otimes \mathcal{C}_{BC}^4$ -state. According to Theorem 1 of Ref. [36] this state is supported on  $\mathcal{C}_A^2 \otimes \mathcal{C}_{BC}^3$ , and must have the form:

$$\rho = \sum_{i=1}^3 |e_{A_i}\rangle\langle e_{A_i}| \otimes |\psi_{BC_i}\rangle\langle \psi_{BC_i}|. \quad (14)$$

We take  $|e\rangle$  orthogonal to  $|e_{A_3}\rangle$ , and demand that  $|f, g\rangle$  is orthogonal to  $|\psi_{BC_1}\rangle$  and  $|\psi_{BC_2}\rangle$ . Setting  $|f_B\rangle = |0\rangle_B + \alpha|1\rangle_B$ , we obtain the following system of linear homogeneous equations for  $|g\rangle$ :

$$(\langle \psi_{BC_i} | 0_B \rangle + \alpha \langle \psi_{BC_i} | 1_B \rangle) |g\rangle = 0 \quad (15)$$

for  $i = 1, 2$ . These equations possess a nontrivial solution if the corresponding determinant of the  $2 \times 2$  matrix vanishes. This leads to a quadratic equation for  $\alpha$ , which has always a solution, and that proves the Corollary.  $\square$

The existence of product vectors in the kernel is used in the proof of the Lemma below. This Lemma provides one of the most important results of this paper: it implies that in  $\mathcal{C}^2 \otimes \mathcal{C}^2 \otimes \mathcal{C}^2$  systems there is no PPT entanglement of rank smaller than 4.

**Lemma 5 :** Any PPT state  $\rho$ , supported on  $\mathcal{C}^2 \otimes \mathcal{C}^2 \otimes \mathcal{C}^2$ , such that  $r(\rho) = 3$ , is separable.

**Proof:** Let  $|e_A, f_B, g_C\rangle$  belongs to  $K(\rho)$ . From the condition  $\rho |e_A, f_B, g_C\rangle = 0$  for the product vector in the kernel, follows that:

$$\begin{aligned} \langle e_A | \rho | \hat{e}_A, f_B, g_C \rangle &= 0, \\ \langle f_B | \rho | e_A, \hat{f}_B, g_C \rangle &= 0, \\ \langle g_C | \rho | e_A, f_B, \hat{g}_C \rangle &= 0. \end{aligned}$$

This means that

$$\begin{aligned} \rho | \hat{e}_A, f_B, g_C \rangle &= | \hat{e}_A \rangle | \psi_{BC} \rangle, \\ \rho | e_A, \hat{f}_B, g_C \rangle &= | \hat{f}_B \rangle | \psi_{AC} \rangle, \\ \rho | e_A, f_B, \hat{g}_C \rangle &= | \hat{g}_C \rangle | \psi_{AB} \rangle. \end{aligned}$$

We define

$$\tilde{\rho} = \rho - \lambda | \hat{e}_A \rangle \langle \hat{e}_A | \otimes | \psi_{BC} \rangle \langle \psi_{BC} |, \quad (16)$$

where  $\lambda = \frac{1}{\langle \hat{e}_A, \psi_{BC} | \rho^{-1} | \hat{e}_A, \psi_{BC} \rangle}$  (see Lemma 2 of Ref. [36]). Now,  $\tilde{\rho}$  is a PPT state with respect to  $A-BC$  partition, i.e.  $\tilde{\rho}^{t_A} \geq 0$ ; this state has the rank  $r(\tilde{\rho}) = 2$ . We rewrite  $\tilde{\rho}$  as:

$$\tilde{\rho} = \lambda_1 | \hat{f}_B \rangle \langle \hat{f}_B | \otimes | \psi_{AC} \rangle \langle \psi_{AC} | + \lambda_2 | \hat{g}_C \rangle \langle \hat{g}_C | \otimes | \psi_{AB} \rangle \langle \psi_{AB} |. \quad (17)$$

We redefine now  $|e_A\rangle = |0\rangle$  and  $|\hat{e}_A\rangle = |1\rangle$ , i.e. change the basis in Alice's system, and represent the vectors  $|\psi_{AC}\rangle$  and  $|\psi_{AB}\rangle$  in the new basis as:

$$|\psi_{AC}\rangle = |0\rangle |\psi_C^1\rangle + |1\rangle |\psi_C^2\rangle, \quad (18)$$

$$|\psi_{AB}\rangle = |0\rangle |\phi_B^1\rangle + |1\rangle |\phi_B^2\rangle. \quad (19)$$

In the matrix form  $\tilde{\rho}$  can be written as:

$$\tilde{\rho} = \left( \begin{pmatrix} \lambda_1 | \hat{f}_B \rangle \langle \hat{f}_B | \otimes | \psi_C^1 \rangle \langle \psi_C^1 | \\ + \lambda_2 | \phi_B^1 \rangle \langle \phi_B^1 | \otimes | \hat{g}_C \rangle \langle \hat{g}_C | \end{pmatrix} \begin{pmatrix} \lambda_1 | \hat{f}_B \rangle \langle \hat{f}_B | \otimes | \psi_C^1 \rangle \langle \psi_C^2 | \\ + \lambda_2 | \phi_B^1 \rangle \langle \phi_B^2 | \otimes | \hat{g}_C \rangle \langle \hat{g}_C | \end{pmatrix} \right) + \left( \begin{pmatrix} \lambda_1 | \hat{f}_B \rangle \langle \hat{f}_B | \otimes | \psi_C^2 \rangle \langle \psi_C^1 | \\ + \lambda_2 | \phi_B^2 \rangle \langle \phi_B^1 | \otimes | \hat{g}_C \rangle \langle \hat{g}_C | \end{pmatrix} \begin{pmatrix} \lambda_1 | \hat{f}_B \rangle \langle \hat{f}_B | \otimes | \psi_C^2 \rangle \langle \psi_C^2 | \\ + \lambda_2 | \phi_B^2 \rangle \langle \phi_B^2 | \otimes | \hat{g}_C \rangle \langle \hat{g}_C | \end{pmatrix} \right).$$

From the positivity of  $\tilde{\rho}$  and  $\tilde{\rho}^{t_A}$  follows that when a diagonal block  $\begin{pmatrix} \lambda_1 | \hat{f}_B \rangle \langle \hat{f}_B | \otimes | \psi_C^2 \rangle \langle \psi_C^2 | \\ + \lambda_2 | \phi_B^2 \rangle \langle \phi_B^2 | \otimes | \hat{g}_C \rangle \langle \hat{g}_C | \end{pmatrix}$  acting on  $|\hat{\phi}_B^2 \hat{\psi}_C^2\rangle$  vanishes, the same must be true for the off-diagonal block in left lower corner. Similarly, the same observation concerns the diagonal  $\begin{pmatrix} \lambda_1 | \hat{f}_B \rangle \langle \hat{f}_B | \otimes | \psi_C^1 \rangle \langle \psi_C^1 | \\ + \lambda_2 | \phi_B^1 \rangle \langle \phi_B^1 | \otimes | \hat{g}_C \rangle \langle \hat{g}_C | \end{pmatrix}$ , the vector  $|\hat{\phi}_B^1 \hat{\psi}_C^1\rangle$ , and the off-diagonal block in the right upper corner. This leads to the system of equations:

$$\langle \hat{f}_B | \hat{\phi}_B^1 \rangle \langle \psi_C^2 | \hat{\psi}_C^1 \rangle = 0, \quad (20)$$

$$\langle \phi_B^2 | \hat{\phi}_B^1 \rangle \langle \hat{g}_C | \hat{\psi}_C^1 \rangle = 0, \quad (21)$$

$$\langle \hat{f}_B | \hat{\phi}_B^2 \rangle \langle \psi_C^1 | \hat{\psi}_C^2 \rangle = 0, \quad (22)$$

$$\langle \phi_B^1 | \hat{\phi}_B^2 \rangle \langle \hat{g}_C | \hat{\psi}_C^2 \rangle = 0. \quad (23)$$

This system of equations implies that at least one of the projectors  $|\psi_{AB}\rangle\langle\psi_{AB}|$  and  $|\psi_{AC}\rangle\langle\psi_{AC}|$  must be a product state. If it is, for instance,  $|\psi_{AB}\rangle\langle\psi_{AB}|$ , then  $|\phi_B^1\rangle = |\phi_B^2\rangle = |\hat{f}_B\rangle$  and  $\rho$  becomes

$$\begin{aligned}\rho &= \lambda_1 |\hat{f}_B\rangle\langle\hat{f}_B| \otimes |\psi_{AC}\rangle\langle\psi_{AC}| \\ &+ \lambda_2 |\tilde{e}_A\rangle\langle\tilde{e}_A| \otimes |\hat{f}_B\rangle\langle\hat{f}_B| \otimes |\hat{g}_C\rangle\langle\hat{g}_C| \\ &+ \lambda |\hat{e}\rangle\langle\hat{e}| \otimes |\psi_{BC}\rangle\langle\psi_{BC}| \\ &= |\hat{f}_B\rangle\langle\hat{f}_B| \otimes \underbrace{(\lambda_1 |\psi_{AC}\rangle\langle\psi_{AC}| + \lambda_2 |\tilde{e}_A\rangle\langle\tilde{e}_A| \otimes |\hat{g}_C\rangle\langle\hat{g}_C|)}_{\sigma} \\ &+ \lambda |\hat{e}\rangle\langle\hat{e}| \otimes |\psi_{BC}\rangle\langle\psi_{BC}|.\end{aligned}\quad (24)$$

The operator  $\sigma$  is a PPT state of rank 2 in  $\mathcal{C}^2 \otimes \mathcal{C}^2$  space of Alice and Charlie. From Peres-Horodecki criterium [9,11] follows that it is separable. The matrix  $\rho$  can thus be written as

$$\begin{aligned}\rho &= \lambda_1 |\tilde{e}_A\rangle\langle\tilde{e}_A| \otimes |\hat{f}_B\rangle\langle\hat{f}_B| \otimes |\tilde{g}_C\rangle\langle\tilde{g}_C| \\ &+ \lambda_2 |\tilde{e}_A\rangle\langle\tilde{e}_A| \otimes |\hat{f}_B\rangle\langle\hat{f}_B| \otimes |\hat{g}_C\rangle\langle\hat{g}_C| \\ &+ \lambda |\hat{e}\rangle\langle\hat{e}| \otimes |\psi_{BC}\rangle\langle\psi_{BC}|.\end{aligned}$$

For the above proof Alice is in no way distinguished. We can also write

$$\begin{aligned}\bar{\rho} &= \rho - \bar{\lambda} |\tilde{e}_A\rangle\langle\tilde{e}_A| \otimes |\hat{f}_B\rangle\langle\hat{f}_B| \otimes |\hat{g}_C\rangle\langle\hat{g}_C| \\ &= \lambda_1 |\tilde{e}_A\rangle\langle\tilde{e}_A| \otimes |\hat{f}_B\rangle\langle\hat{f}_B| \otimes |\tilde{g}_C\rangle\langle\tilde{g}_C| \\ &+ \lambda |\hat{e}_A\rangle\langle\hat{e}_A| \otimes |\psi_{BC}\rangle\langle\psi_{BC}|,\end{aligned}$$

where  $\bar{\lambda} \equiv \lambda_2 = \frac{1}{\langle \tilde{e}_A \hat{f}_B \hat{g}_C | \rho^{-1} | \tilde{e}_A \hat{f}_B \hat{g}_C \rangle}$ , and  $\bar{\rho}$  is a PPT state with respect to  $C-AB$  partition. The projection of  $\bar{\rho}$  onto  $|\tilde{e}_A\rangle$  gives  $\langle \tilde{e}_A | \bar{\rho} | \tilde{e}_A \rangle \sim |\psi_{BC}\rangle\langle\psi_{BC}|$ . This means, however, that  $|\psi_{BC}\rangle$  must be a product vector, and that concludes the proof.  $\square$

### C. Separability of states of rank $N$ in $\mathcal{C}^2 \otimes \mathcal{C}^2 \otimes \mathcal{C}^N$ systems

Now we are in the position to prove the main theorem of this section. Before that we have to complete, however, the discussion of the case  $N = 3$ . To this aim we prove the following Lemma:

**Lemma 6 :** *Any PPT state  $\rho$ , supported on  $\mathcal{C}^2 \otimes \mathcal{C}^2 \otimes \mathcal{C}^3$ , such that  $r(\rho) = 3$ , is separable.*

**Proof:** We consider the system  $\mathcal{C}^2 \otimes \mathcal{C}^2 \otimes \mathcal{C}^3$ -System as a  $\mathcal{C}_{AB}^4 \otimes \mathcal{C}_C^3$  system. According to the Theorem (1) three possibilities may occur:

- The state is supported on  $\mathcal{C}_{AB}^3 \otimes \mathcal{C}_C^3$ . Then the density matrix must have a form

$$\begin{aligned}\rho &= \Lambda_1 |e_{AB_1}\rangle\langle e_{AB_1}| \otimes |f_{C_1}\rangle\langle f_{C_1}| \\ &+ \Lambda_2 |e_{AB_2}\rangle\langle e_{AB_2}| \otimes |f_{C_2}\rangle\langle f_{C_2}| \\ &+ \Lambda_3 |e_{AB_3}\rangle\langle e_{AB_3}| \otimes |f_{C_3}\rangle\langle f_{C_3}|.\end{aligned}$$

Since the vectors  $|f_{C_i}\rangle$  are linearly independent, we can find a vector  $|C\rangle$  in Charlie's system such that  $\langle C | \rho | C \rangle \sim |e_{AB_1}\rangle\langle e_{AB_1}|$ . Because the considered state has the PPT property with respect to all partition, the projected state  $|e_{AB_1}\rangle\langle e_{AB_1}|$  is also PPT, and as such must be a product state.

- The state is supported on  $\mathcal{C}_{AB}^2 \otimes \mathcal{C}_C^3$ . The same method of projecting onto appropriately chosen vector in Charlie's space allows to prove the separability.
- The state is supported  $\mathcal{C}_{AB}^3 \otimes \mathcal{C}_C^2$ . That is, however, nothing else but a state in  $\mathcal{C}^2 \otimes \mathcal{C}^2 \otimes \mathcal{C}^2$  system with rank 3. Its separability follows from Lemma 5.

This concludes the proof of the Lemma 6.  $\square$

Now, all the above presented results can be brought together in a form of the following theorem:

**Theorem 2 :** *Every PPT state, supported on  $\mathcal{C}^2 \otimes \mathcal{C}^2 \otimes \mathcal{C}^N$ , such that  $r(\rho) = N$ , is separable and has the canonical form of the Lemma 1,*

$$\begin{aligned}\rho &= \sqrt{D} \begin{pmatrix} B^\dagger C^\dagger C B & B^\dagger C^\dagger C & B^\dagger C^\dagger B & B^\dagger C^\dagger \\ C^\dagger C B & C^\dagger C & C^\dagger B & C^\dagger \\ B^\dagger C B & B^\dagger C & B^\dagger B & B^\dagger \\ C B & C & B & 1 \end{pmatrix} \sqrt{D} \quad (25) \\ &= \sqrt{D} \begin{pmatrix} B^\dagger C^\dagger \\ C^\dagger \\ B^\dagger \\ 1 \end{pmatrix} (C B \ C \ B \ 1) \sqrt{D}, \quad (26)\end{aligned}$$

where  $B, C$  and  $D$  are operators acting in Charlie's space that fulfill  $[B, B^\dagger] = [C, C^\dagger] = [C, B] = [C, B^\dagger] = 0$  and  $D = D^\dagger$ .

In the next section we will study states in  $\mathcal{C}^2 \otimes \mathcal{C}^2 \otimes \mathcal{C}^N$  with low ranges, but  $\geq N$ . By looking at product vectors in the ranges of  $\rho$  and its partial transposes it is possible to check separability for low rank matrices, similarly as in the case of bipartite systems in  $\mathcal{C}^M \otimes \mathcal{C}^N$  [36,37].

### III. SEPARABILITY CHECKS AND CRITERIA FOR GENERIC LOW RANK STATES IN $\mathcal{C}^2 \otimes \mathcal{C}^2 \otimes \mathcal{C}^N$ SYSTEMS

In this section we will study the PPT states  $\rho$  that posses a finite number of product vectors in their range  $|e_i, f_i, g_i\rangle \in \mathcal{C}^2 \otimes \mathcal{C}^2 \otimes \mathcal{C}^N$  such that  $|e_{A_i}, f_{B_i}, g_{C_i}\rangle \in R(\rho)$ ,  $|e_{A_i}^*, f_{B_i}, g_{C_i}\rangle \in R(\rho^{t_A})$ ,  $|e_{A_i}, f_{B_i}^*, g_{C_i}\rangle \in R(\rho^{t_B})$  and  $|e_{A_i}^*, f_{B_i}^*, g_{C_i}\rangle \in R(\rho^{t_{AB}})$ . We will show that this is generically the case when  $r(\rho) + r(\rho^{t_A}) + r(\rho^{t_B}) + r(\rho^{t_{AB}}) \leq 15N - 1$ . Let us call the set of such vectors  $V[\rho]$ . The search for the desired product vectors  $\{|e_{A_i}, f_{B_i}, g_{C_i}\rangle\} \in \mathcal{C}^2 \otimes \mathcal{C}^2 \otimes \mathcal{C}^N$  is reduced to the problem of solving a system of multipolynomial equations [36]. When the number of equations is equal to (bigger than) the number

of available unknown parameters, one expect the number of solutions to be finite (zero). The states of low ranks fulfilling this property has been termed *generic* in Ref. [37]. In particular, the states for which the number of the desired vectors in any of the considered ranges is smaller than the corresponding rank, must be entangled. Particularly important are states that do not contain any product vector of the above described properties in the range. Such states are termed *edge states*, and play major role in characterization and classification of the PPT entangled states [33].

### A. Generic states

Let  $|K_i\rangle$ ,  $|K_{A_i}\rangle$ ,  $|K_{B_i}\rangle$  and  $|K_{AB_i}\rangle$  are linearly independent vector that span the kernels of  $\rho$ ,  $\rho^{t_A}$ ,  $\rho^{t_B}$  and  $\rho^{t_{AB}}$ , respectively, so that:

$$\begin{aligned} K(\rho) &= \text{span}\{|K_i\rangle, i = 1, \dots, k(\rho)\}, \\ K(\rho^{t_A}) &= \text{span}\{|K_{A_i}\rangle, i = 1, \dots, k(\rho^{t_A})\}, \\ K(\rho^{t_B}) &= \text{span}\{|K_{B_i}\rangle, i = 1, \dots, k(\rho^{t_B})\}, \\ K(\rho^{t_{AB}}) &= \text{span}\{|K_{AB_i}\rangle, i = 1, \dots, k(\rho^{t_{AB}})\}. \end{aligned}$$

Choosing an orthonormal basis in Alice's and Bob's space we can write those vectors as:

$$\begin{aligned} |K_i\rangle &= |00\rangle|k_i^{00}\rangle + |01\rangle|k_i^{01}\rangle + |10\rangle|k_i^{10}\rangle + |11\rangle|k_i^{11}\rangle \\ |K_{A_i}\rangle &= |00\rangle|k_{A_i}^{00}\rangle + |01\rangle|k_{A_i}^{01}\rangle + |10\rangle|k_{A_i}^{10}\rangle + |11\rangle|k_{A_i}^{11}\rangle \\ |K_{B_i}\rangle &= |00\rangle|k_{B_i}^{00}\rangle + |01\rangle|k_{B_i}^{01}\rangle + |10\rangle|k_{B_i}^{10}\rangle + |11\rangle|k_{B_i}^{11}\rangle \\ |K_{AB_i}\rangle &= |00\rangle|k_{AB_i}^{00}\rangle + |01\rangle|k_{AB_i}^{01}\rangle + |10\rangle|k_{AB_i}^{10}\rangle + |11\rangle|k_{AB_i}^{11}\rangle. \end{aligned}$$

A product vector in  $|e, f, g\rangle \in V[\rho]$  has the property that it and its partial complex conjugates have to be orthogonal to the corresponding kernels, i.e.:

$$\begin{aligned} \langle K_i | e_A, f_B, g_C \rangle &= 0, \\ \langle K_{A_i} | e_A^*, f_B, g_C \rangle &= 0, \\ \langle K_{B_i} | e_A, f_B^*, g_C \rangle &= 0, \\ \langle K_{AB_i} | e_A^*, f_B^*, g_C \rangle &= 0. \end{aligned} \quad (27)$$

We expand now  $|e_A, f_B, g_C\rangle$  in the local basis of Alice and Bob:

$$\begin{aligned} |e_A, f_B, g_C\rangle &= (\alpha|0\rangle + |\beta\rangle) \otimes (\beta|0\rangle + |\gamma\rangle) \otimes |g\rangle \\ &= (\alpha\beta|00\rangle + \alpha|01\rangle + \beta|10\rangle + |\gamma\rangle) \otimes |g\rangle. \end{aligned}$$

We observe that Eqs. (27) can be rewritten as :

$$A(\alpha, \beta; \alpha^*, \beta^*)|g\rangle = 0, \quad (28)$$

where  $A(\alpha, \beta; \alpha^*, \beta^*)$  is a  $(k(\rho) + k(\rho^{t_A}) + k(\rho^{t_B}) + k(\rho^{t_{AB}})) \times N$  matrix, which reads:

$$A(\alpha, \beta; \alpha^*, \beta^*) = \begin{pmatrix} \alpha\beta\langle k_i^{00} | + \alpha\langle k_i^{01} | + \beta\langle k_i^{10} | + \langle k_i^{11} | \\ \alpha^*\beta\langle k_{A_i}^{00} | + \alpha^*\langle k_{A_i}^{01} | + \beta\langle k_{A_i}^{10} | + \langle k_{A_i}^{11} | \\ \alpha\beta^*\langle k_{B_i}^{00} | + \alpha\langle k_{B_i}^{01} | + \beta^*\langle k_{B_i}^{10} | + \langle k_{B_i}^{11} | \\ \alpha^*\beta^*\langle k_{AB_i}^{00} | + \alpha^*\langle k_{AB_i}^{01} | + \beta^*\langle k_{AB_i}^{10} | + \langle k_{AB_i}^{11} | \end{pmatrix}.$$

Eqs. (27) have a nontrivial solution with  $|e\rangle \neq 0, |f\rangle \neq 0$  and  $|g\rangle \neq 0$  iff the rank of  $A$  is smaller than  $N$ . That implies that at most  $N - 1$  rows of the matrix  $A$  are linearly independent. That means that  $(k(\rho) + k(\rho^{t_A}) + k(\rho^{t_B}) + k(\rho^{t_{AB}})) - N + 1$  minors of dimension  $N \times N$  of the matrix  $A$  must vanish.

Let us consider the marginal case, when  $k(\rho) + k(\rho^{t_A}) + k(\rho^{t_B}) + k(\rho^{t_{AB}}) = 2 + (N - 1)$ . In this case we combine the first  $N - 1$  rows with the remaining two and obtain exactly two different minors, and thus two equations for complex  $\alpha, \beta$ , or more precisely four real equation for real and imaginary parts of  $\alpha, \beta$ . Such equations generically will have a finite number of solutions. The case when  $k(\rho) + k(\rho^{t_A}) + k(\rho^{t_B}) + k(\rho^{t_{AB}}) > 2 + (N - 1)$ , i.e.

$$(r(\rho) + r(\rho^{t_A}) + r(\rho^{t_B}) + r(\rho^{t_{AB}})) < 15N - 1 \quad (29)$$

means that we have more equations than parameters, and generically there will be no solution, or at least the number of solutions will be even more limited than in the marginal case. The PPT states fulfilling the inequality (29) are generically the edge states, provided their rank and/or the ranks of their partial transposes are greater than  $N$ , since otherwise the Theorem of the previous section would apply. Conversely, if

$$(r(\rho) + r(\rho^{t_A}) + r(\rho^{t_B}) + r(\rho^{t_{AB}})) > 15N - 1, \quad (30)$$

then the matrix  $A$  has less equal than  $N$  rows, and one can always use the freedom of parameters to find a solution, and subtract a projector onto a product vector from  $\rho$  keeping its positivity and PPT property intact.

In the following we will concentrate ourselves on the case  $k(\rho) + k(\rho^{t_A}) + k(\rho^{t_B}) + k(\rho^{t_{AB}}) \geq N + 1$ , for which the number of solutions is expected to be finite. Such states will be called as in Ref. [37] generic. For those states it is simple to check the separability, similarly as discussed in Ref. [36,37]. The check is easy, because we know that if the considered state is separable, then it is represented as a convex sum of projectors on the vectors from the set  $V[\rho]$ , and the latter has a *finite* cardinality. We will discuss this in more detail for the case of  $\mathcal{C}^2 \otimes \mathcal{C}^2 \otimes \mathcal{C}^2$  systems.

## IV. SEPARABILITY CHECKS AND CRITERIA FOR GENERIC LOW RANK PPT STATES IN $\mathcal{C}^2 \otimes \mathcal{C}^2 \otimes \mathcal{C}^2$ SYSTEMS

As a special, but important example we consider the case of PPT states in  $\mathcal{C}^2 \otimes \mathcal{C}^2 \otimes \mathcal{C}^2$  states (3 qubit systems). We will use here the results of the previous sections. The 3 qubit case is particularly interesting as a first step toward multiple entangled systems, providing a challenge for both the theory and experiment.

Generically, if

$$(r(\rho) + r(\rho^{t_A}) + r(\rho^{t_B}) + r(\rho^{t_{AB}})) \leq 28, \quad (31)$$

then the set  $V[\rho]$  is empty and the state  $\rho$  is a PPT entangled edge state, provided all the ranks are greater 3, since otherwise the Lemma 5 of the previous section applies. We discuss the different cases below

#### A. The case $r(\rho) = 2, 3$

From the results of the previous section we know that such PPT states are separable.

#### B. The case $r(\rho) = 4$

The state of rank 4 in a 3 qubit system may be regarded a state in  $\mathcal{C}^2 \otimes \mathcal{C}^4$  of rank 4. From the Theorem 1 that this state is bipartite separable, and moreover has a unique decomposition into a sum of four projectors on product (biseparable) vectors in  $\mathcal{C}_A^2 \otimes \mathcal{C}_{BC}^4$ . From uniqueness, we gather that  $\rho$  is then separable iff the product vectors in this decomposition are completely separable, i.e. are product vectors in  $\mathcal{C}^2 \otimes \mathcal{C}^2 \otimes \mathcal{C}^2$ . Otherwise, the state is entangled, although biseparable. In fact it must be biseparable with respect to all partitions, i.e. also  $\mathcal{C}_B^2 \otimes \mathcal{C}_{AC}^4$  and  $\mathcal{C}_C^2 \otimes \mathcal{C}_{AB}^4$ . Examples of such states are known, in particular those are the state constructed from unextendible product basis [26].

#### C. The case $r(\rho) = r(\rho^{t_A}) = 5$

This case is also easy because first of all the bipartite separability with respect to the partition  $A - BC$  has to be checked. As shown in Ref. [36], a PPT state in  $\mathcal{C}_A^2 \otimes \mathcal{C}_{BC}^4$  is (bipartite) separable, iff the set of bipartite product vectors  $V_{A-BC}[\rho]$ , corresponding to the partition  $A - BC$  is not empty. In the present case it must not only contain a bipartite product vector, but a tripartite product vector. If such vectors exist, generically there will be finite number of them, and at least 5 of them must belong to the set  $V_{A-B-C}[\rho]$ .

#### D. The case $r(\rho) + r(\rho^{t_A}) + r(\rho^{t_B}) + r(\rho^{t_{AB}}) \leq 28$

In this case we have more equations than available parameters, and we expect that the set  $V[\rho]$  will be empty, whereas the state  $\rho$  will be an edge state. If this is not the case, we expect first of all that there is a finite number of product vectors in  $V[\rho]$ . Thus, checking if  $\rho$  can be represented as a convex sum of projectors onto the elements of  $V[\rho]$  can be performed exactly using the same methods as discussed in Ref. [36].

#### E. The case $r(\rho) = r(\rho^{t_A}) = r(\rho^{t_B}) = r(\rho^{t_{AB}}) = 7$

If  $V[\rho]$  is empty, this case describes an example of an edge state with maximal sum of ranks. Such an example has been constructed in Ref. [32]. Let us estimate how many elements can the set  $V[\rho]$  contain maximally. To this aim we write the matrix  $A$ :

$$A(\alpha, \beta; \alpha^*, \beta^*) = \begin{pmatrix} \alpha\beta\langle k_{00}^{00} | + \alpha\langle k_{01}^{01} | + \beta\langle k_{10}^{10} | + \langle k_{11}^{11} | \\ \alpha^*\beta\langle k_{00}^{01} | + \alpha^*\langle k_{01}^{01} | + \beta\langle k_{10}^{10} | + \langle k_{11}^{11} | \\ \alpha\beta^*\langle k_{00}^{10} | + \alpha\langle k_{01}^{10} | + \beta^*\langle k_{10}^{10} | + \langle k_{11}^{11} | \\ \alpha^*\beta^*\langle k_{00}^{11} | + \alpha^*\langle k_{01}^{11} | + \beta^*\langle k_{10}^{11} | + \langle k_{11}^{11} | \end{pmatrix}$$

Let us denote a polynomial  $P$  of orders  $X$  and  $Y$  in variable  $z$  and  $z^*$  by  $P_{X,Y}(z)$ . Combining the first and the third row, and the second and the fourth row of  $A$  we obtain the minors of the form:

$$\alpha^2 P_{(1,1)}^{(1)}(\beta) + \alpha P_{(1,1)}^{(2)}(\beta) + P_{(1,1)}^{(3)}(\beta) = 0. \quad (32)$$

$$(\alpha^*)^2 R_{(1,1)}^{(1)}(\beta) + \alpha^* R_{(1,1)}^{(2)}(\beta) + R_{(1,1)}^{(3)}(\beta) = 0. \quad (33)$$

The remaining four combinations of rows give us

$$\alpha\alpha^* Q_{(2,0)}^{(1)}(\beta) + \alpha Q_{(2,0)}^{(2)}(\beta) + \alpha^* Q_{(2,0)}^{(3)}(\beta) + Q_{(2,0)}^{(4)}(\beta) = 0 \quad (34)$$

$$\alpha\alpha^* Q_{(0,2)}^{(1)}(\beta) + \alpha Q_{(0,2)}^{(2)}(\beta) + \alpha^* Q_{(0,2)}^{(3)}(\beta) + Q_{(0,2)}^{(4)}(\beta) = 0, \quad (35)$$

and two equations of the form

$$\alpha\alpha^* Q_{(1,1)}^{(1)}(\beta) + \alpha Q_{(1,1)}^{(2)}(\beta) + \alpha^* Q_{(1,1)}^{(3)}(\beta) + Q_{(0,1)}^{(4)}(\beta) = 0. \quad (36)$$

Only 3 of the above equations are independent, but we have to our three complex conjugated equations to our disposal, and in particular the conjugate of Eq. (33),

$$\alpha^2 R_{(1,1)}^{(1)*}(\beta) + \alpha R_{(1,1)}^{(2)*}(\beta) + R_{(1,1)}^{(3)*}(\beta) = 0. \quad (37)$$

A good strategy is to multiply Eq. (32) by  $R_{(1,1)}^{(1)*}(\beta)$ , and Eq. (37) by  $P_{(1,1)}^{(1)}(\beta)$ , and subtract one from another in order to obtain

$$\alpha = T_{(2,2)}^{(1)}(\beta) / T_{(2,2)}^{(1)}(\beta). \quad (38)$$

Inserting this solution into Eq. (32) we obtain a polynomial of orders 5, 5 in  $\beta$  and  $\beta^*$ . Another independent polynomial is obtained by complex conjugation. The variables  $\beta$  and  $\beta^*$  are then treated as independent ones, similarly in the Appendices of Ref. [36]. According to the result presented there, a system of two polynomial equations of order  $X, Y$  with  $X \leq Y$  for two variables  $\beta$  and, say,  $\bar{\beta}$  has at most  $2^{X,Y}$  solutions for  $\beta$ . In the present case we expect thus that the number of solutions is  $\leq 160$ . Most of these solutions will have to be rejected typically, since they do not fulfill the conditions Eqs. (34)-(36).

#### F. The case $r(\rho) + r(\rho^{t_A}) + r(\rho^{t_B}) + r(\rho^{t_{AB}}) = 29$

This is a marginal case in which the number of equations is equal to the number of parameters, so that generically we have a finite number of product vectors in  $V[\rho]$ ,



and a possibility of performing the relatively straightforward separability check. For example, if we consider  $r(\rho) = r(\rho^{t_A}) = r(\rho^{t_{AB}}) = 7$  and  $r(\rho^{t_B}) = 8$ . In this case only two minors are independent, and we have, for instance, to solve Eq. (34), one of the Eqs. (36), and their complex conjugates. By multiplying Eqs. (34) and (36) and its complex conjugates them by appropriate polynomials in  $\beta, \beta^*$ , and subtracting one from another we obtain two linear equations for  $\alpha, \alpha^*$  of the form

$$\alpha S_{(3,1)}^{(1)}(\beta) + \alpha^* S_{(3,1)}^{(2)}(\beta) + S_{(3,1)}^{(3)}(\beta) = 0. \quad (39)$$

and the complex conjugate of the above Eq. (39). This system of two linear equations can be solved so that we obtain

$$\alpha = T_{(4,4)}^{(1)}(\beta) / T_{(4,4)}^{(1)}(\beta). \quad (40)$$

Inserting this solution into Eq. (36) and obtain in this way a polynomial of order 9 in  $\beta$  and  $\beta^*$ . Another independent polynomial is obtained by complex conjugating Eq. (34). The variables  $\beta$  and  $\beta^*$  are then treated as independent ones, similarly as discussed in the Appendix of Ref. [36]. According to Ref. [36] we expect in this case maximally  $2^9 \times 9 = 4608$  solutions for  $\beta$ .

### G. Canonical form of non-decomposable entanglement witnesses

For completeness it is worth mentioning that it is possible to generalize the results of Ref. [33] to case of 3 qubit systems (and in general in tripartite systems). Let us remind the readers that an entanglement witness is a hermitian operator  $W$ , for which  $\text{Tr}(W\sigma) \geq 0$  for any separable state  $\sigma$ , whereas  $\text{Tr}(W\rho) < 0$  for some entangled state  $\rho$ . We say that  $W$  detects then  $\rho$ . A non-decomposable witness is a witness that detects a PPT entangled state. Using exactly the same arguments as in Ref. [33] one shows that a non-decomposable entanglement witness must have the canonical form

$$W = P + Q^{t_A} + R^{t_B} + S^{t_{AB}} - \epsilon \mathbf{1}, \quad (41)$$

where

$$\epsilon = \inf_{|e,f,g\rangle} \langle e, f, g | P + Q^{t_A} + R^{t_B} + S^{t_{AB}} | e, f, g \rangle, \quad (42)$$

the operators  $P, Q, R, S$  are positively definite,  $R(P) = K(\delta)$ ,  $R(Q) = K(\delta^{t_A})$ ,  $R(R) = K(\delta^{t_B})$ ,  $R(S) = K(\delta^{t_{AB}})$ , and  $\delta$  is an edge state, i.e. such state for which by definition the set  $V[\delta]$  is empty, which implies automatically that  $\epsilon$  is strictly positive. According to the results of this section, in three qubit system, the state  $\delta$  is a generic state with  $r(\delta) + r(\delta^{t_A}) + r(\delta^{t_B}) + r(\delta^{t_{AB}}) \leq 28$ .

## V. CONCLUSIONS

We have generalized previously obtained results for PPT state in  $\mathcal{C}^2 \otimes \mathcal{C}^N$  and  $\mathcal{C}^M \otimes \mathcal{C}^N$  system to PPT states in  $\mathcal{C}^2 \otimes \mathcal{C}^2 \otimes \mathcal{C}^N$ . We have developed a method of "local projections" together with the PPT property to prove separability of low rank states and to obtain separability criteria for low rank states. These methods together with methods developed in Refs. [36,37] provide very general mathematical tools to study separability and entanglement in multipartite systems.

The main results of this paper are:

- The proof that all states with positive partial transposes that have rank  $\leq N$  are separable, and have a certain canonical form;
- The proof that for the 3 qubit case ( $N = 2$ ) all PPT states  $\rho$  that have rank 3 are separable;
- The presentation of constructive separability checks for the states  $\rho$  that have the sum of the rank of  $\rho$  and the ranks of partial transposes with respect to all subsystems smaller than  $15N - 1$ .
- The detailed discussion of the above mentioned constructive separability checks for the case  $N = 2$ ;
- Presentation of the canonical form of non-decomposable entanglement witnesses in 3 qubit systems.

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